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Front propagation under periodic forcing in reaction-diffusion systems

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Abstract. One- and two-component bistable reaction-diffusion systems under external force are considered. The simplest case of a periodic forcing of cosine type is chosen. Exact analytical solutions for the traveling fronts are obtained for a piecewise linear approximation of the non-linear reaction term. Velocity equations are derived from the matching conditions. It is found that in the presence of forcing there exists a set of front solutions with different phases (matching point coordinates ξ_0) leading to velocity dependencies on the wavenumber that are either monotonic or oscillating. The general characteristic feature is that the nonmoving front becomes movable under forcing. However, for a specific choice of wavenumber and phase, there is a nonmoving front at any value of the forcing amplitude. When the forcing amplitude is large enough, the velocity bifurcates to form two counterpropagating fronts. The phase portraits of specific types of solutions are shown and briefly discussed.

PACS. 05.45.-a Nonlinear dynamics and nonlinear dynamical systems – 47.20.Ma Interfacial instability – 47.54.+r Pattern selection; pattern formation

1 Introduction

Reaction-diffusion equations have become a prototype for describing propagating wave behavior – from chemical waves to biological populations. Wave propagation in these systems can be effectively controlled by application of an external forcing [1–4]. This forcing can be prescribed *a priori* (*i.e.*, as a periodic modulation of excitability [2,3]) or computed on-line using the data of the momentary state of the medium by closing a feedback loop [4,5]. The phenomenology of this situation is well known. The properties of an external forcing can be studied experimentally by using the light-sensitive Belousov-Zhabotinsky reaction for which the absorption of transmitted light depends on the concentration of chemical species [5].

Having in mind that a general irregular forcing may be represented *via* Fourier decomposition as a superposition of harmonically oscillating "modes", it is instructive to investigate the wave behaviour under a periodically oscillating forcing. The case of a force oscillating with time was examined in reference [2,3,6]. A "pulling effect" [6] of the fronts was found: it was shown that the mean velocity of the perturbed front is increased as compared to that of the unperturbed front. Effectively, the case of time-dependent forcing describes a time-dependent excitability, *i.e.*, a null-cline with periodically oscillating constants. In our paper, we study another interesting case of periodic forcing, namely one that is nonmoving in the comoving frame, *i.e.* traveling with the wave. This case differs in significant ways from parameter-dependent (time or spatial) forcings. The problem becomes an inhomogeneous one and the general solution acquires an additive part. When the wave speed is equal to zero the system degenerates into the spatially forced one.

The basic evolution equations that describe the simplest patterned structures (fronts and pulses) are nonlinear PDEs of the parabolic type[1]. We will consider here only front solutions (the pulse solutions will be studied elsewhere). There is a fundamental difference between the front propagation into a meta-stable state and into an unstable state. In the first case the front has a unique velocity. In the second case, there is a continuum of possible velocities [9]. An example of front propagation into an unstable state is exhibited by the Fisher equation with a quadratic nonlinear reaction term. In the case of front propagation into a meta-stable state, the reaction-diffusion equation has a cubic nonlinearity and corresponds to a bistable model.

Our approach will be analytical rather than numerical. This is possible because the nonlinear reaction term is a piecewise linear function. Piecewise linear approximations of the nonlinear term have been employed in a number of situations [9–11] and have the advantage of being able to

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[1] Also there are models based on the PDEs of the hyperbolic type [7,8].

reduce existence problems for traveling waves to root finding for certain nonlinear algebraic equations. The aim of the present paper is the derivation of analytical solutions for the front propagating under external forcing and the determination of the velocity. We present here results that have been obtained in the most general case of harmonically oscillating forcing and consider both one-component and two-component models, as it is common knowledge that two variables are necessary for the oscillatory and bistable dynamics in the Belousov-Zhabotinsky reaction. In the present report we restrict ourselves to N-systems (bistable models) which are characterized by an inverted N-shaped reaction term dependence with three zeros. Evidently, the systems of the considered type are bistable and the front solution connecting these two states is propagating into a meta-stable state.

2 One-component model

We use in our studies a general mathematical model describing bistable media in terms of a reaction-diffusion equation, $\partial u/\partial t = f(u) + \partial^2 u/\partial x^2$, where the variable $u(x,t)$ represents the concentration when the model describes a chemical reaction and the rate function (reaction term) $f(u)$ characterizes the nonlinearity of the system and has an inverted N-shaped profile. We use here a piecewise linear approximation of this term, consisting of two shifted linear pieces. To introduce a forcing $\bar{f}(x,t)$, the equation is slightly modified as

$$\frac{\partial u}{\partial t} = -u - 1 + 2\theta(u - u_0) + \bar{f}(x,t) + \frac{\partial^2 u}{\partial x^2}\,, \qquad (1)$$

where $\bar{f}(x,t)$ is the external forcing and $\theta(u-u_0)$ is the Heaviside function, u_0 is the discontinuity point of the null-cline $f(u) = 0$ (which would correspond to a point of inflection in a model with a continuous nonlinear null-cline). The aim of our considerations is a traveling wave solution. So, we introduce the traveling frame coordinate $\xi = x - ct$, where c is the front velocity, and rewrite (1) in the form of traveling wave equation

$$u_{\xi\xi} + cu_\xi - u - 1 + 2\theta(u - u_0) + \bar{f}(\xi) = 0\,, \qquad (2)$$

where the subscripts on u denote derivatives. Here we consider the case when the forcing $\bar{f}$ is *nonmoving in the comoving frame*, *i.e.*, it is a function of only ξ. The simplest case of the periodic forcing $\bar{f}(\xi)$ may be presented by the following expression: $\bar{f}(\xi) = h\cos(k\xi)$. It is not necessary for us to assume in the following that the oscillations of the forcing are slow and the considered fronts may be described to the needed accuracy within the adiabatic approximation [6]. We will find the wave solution exactly.

The simplest particular case, the front wave, will be our initial concern. Due to the two-piece structure of the reaction term $f(u)$, our solution must also consist of two pieces. To construct the front solutions from two pieces $u_{1,2}(\xi)$, we impose the boundary conditions at infinity and the matching conditions for functions and their derivatives at some matching point $\xi = \xi_0$, where the two parts of the solution are patched together. An additional (third) equation is obtained using the fact that we know the u value of the matching point.

First, we write the general solution

$$u(\xi) = \sum_{i=1}^{2} A_i \mathrm{e}^{\lambda_i \xi} + \bar{u}(\xi) + u^*\,, \quad u^* = \text{const.}\,, \qquad (3)$$

where the A_i are constants to be determined in each of the regions $u \le u_0$ and $u \ge u_0$; $\bar{u}(\xi)$ is a particular solution of the inhomogeneous equation; u^* is the value of the field at the two fixed points of the system to which the solution $u(\xi)$ without $\bar{u}(\xi)$ must tend for $\xi \to \pm\infty$ and $\lambda_{1,2} = -c/2 \pm \sqrt{c^2/4 + 1}$ are the eigenvalues of the homogeneous problem. We see that λ_1 is positive and λ_2 is negative. Hence the two-piece solution reads

$$\begin{aligned} u_1(\xi) &= A_1 \mathrm{e}^{\lambda_1 \xi} + \bar{u}(\xi) - 1\,, \quad \xi \le \xi_0\,, \\ u_2(\xi) &= A_2 \mathrm{e}^{\lambda_2 \xi} + \bar{u}(\xi) + 1\,, \quad \xi \ge \xi_0\,, \end{aligned} \qquad (4)$$

where $\bar{u}(\xi) = R\cos(k\xi) + Q\sin(k\xi)$ and $R, Q = \text{const.}$ To determine the constants R and Q we insert $\bar{u}(\xi)$ into (2) and collect the terms multiplied by $\cos(k\xi)$ and $\sin(k\xi)$. The result is

$$R = h\frac{k^2+1}{(k^2+1)^2 + c^2k^2}\,, \quad Q = -h\frac{ck}{(k^2+1)^2 + c^2k^2}\cdot \qquad (5)$$

Hence the constant R is always positive when $h > 0$ and negative when $h < 0$. The sign of Q depends on the combination of parameters hck and Q vanishes when one of these parameters is equal to zero. In the case of constant forcing ($k = 0$), the particular solution reduces to a constant with $R = h$ and $Q = 0$.

The unknown constants $A_{1,2}$ may be determined and explicitly expressed as functions of the null-cline parameter u_0 and the front velocity c from the matching conditions. From these conditions we can also obtain the front velocity the same way as in previous work [12], first reducing the number of equations from three to one by expressing the constants of integration in terms of the as yet unknown velocity. Then we obtain the velocity equation. In previous works, we chose the matching point value ξ_0 equal to zero. However now, in the presence of the ξ-dependent forcing, the translation invariance of the model equation is violated and the magnitude of the matching point coordinate ξ_0 cannot be chosen arbitrarily or rather, the front solution depends on this value ξ_0, *i.e.*, we have a family of front solutions with different ξ_0. The form of the two-piece solution (4) remains the same, the ξ_0-dependence appears in the velocity equation

$$c = [u_0 - R\cos(k\xi_0) - Q\sin(k\xi_0)](\lambda_1 - \lambda_2)\,. \qquad (6)$$

We see that the ξ_0-dependence is present only in the forcing terms (R, Q-terms). The exponential terms with ξ_0 which arose in the matching equations were eliminated during the reduction procedure. From the velocity equation (6) it follows also that the constant ($k = 0$) external forcing acts on the velocity behaviour in the same way as a non-zero value of u_0. Also, in this case, translational invariance is restored. Therefore, we may put $u_0 = 0$ for simplicity and just consider the influence of a truly oscillatory forcing.

Let us consider a set of possible solutions. It is instructive to deal with some particular cases. A first particular case (case I) is when $k\xi_0 = 2\pi n, n = 0, \pm 1, \pm 2, ...$ [2]. In this case the Q-term vanishes and we obtain

$$c + R(\lambda_1 - \lambda_2) = 0. \quad (7)$$

As $\lambda_1 \neq \lambda_2$, the velocity is equal to zero only when $R = 0$, *i.e.*, there is no nonmoving solution (stationary front) in the presence of forcing. Using the expressions for R and λ_i, the velocity equation can be rewritten as

$$c[c^2k^2 + (k^2+1)^2] + h(k^2+1)\sqrt{c^2+4} = 0\,. \quad (8)$$

This equation has only one real solution, both other roots are imaginary. The equation remains the same under a change of sign of k. This is not surprising, because the R-term is coupled with the cosine function. In the case of constant forcing ($k = 0$), the velocity equation reduces to $c + h\sqrt{c^2+4} = 0$ which has only a real root at $|h| < 1$. By taking the square of the reduced equation, we obtain $c^2/4 = h^2/(1-h^2)$ and then the restriction $|h| < 1$ (which is mathematically evident) may be explained in physical terms using the fact that the constant external forcing acts on the velocity the same way as a shifted value of the discontinuity point u_0. Recall that without forcing but an asymmetric null-cline the velocity equation remains the same on replacing $h \to -u_0$ (see Eq. (2.2.13) in [9]). In this system, the condition $|u_0| < 1$ means that the u_0-point (the point at which there is a jump discontinuity of the null-cline) lies within the interval between the two fixed points, *i.e.*, this restriction represents an *existence condition* for front solutions. The non-zero value of velocity at constant forcing is understood by considering the null-cline $f(u) + \bar{f} = -u \mp 1 + \bar{f} = 0$. The introduction of the external forcing produces a shift along the $f(u)$-axis, changing the symmetric null-cline $f(u) = -u \mp 1 = 0$ into an asymmetric one. Then the propagating front has non-zero velocity. When $u_0 = 0$, the front propagates always at non-zero velocity at $h \neq 0$ and when $u_0 \neq 0$, there are values of h and u_0 for which the front velocity is equal to zero. From equation (8), it also follows that under external forcing a nonmoving front becomes movable; the velocity is equal to zero only when $h = 0$. The sign of the velocity is determined by the sign of the parameter h: when h is positive, the velocity is negative and *vice versa*.

The velocity diagrams $c = c(h)$, k fixed, and $c = c(k)$, h fixed, at $u_0 = 0$ (which corresponds to a function $f(u)$ with inversion symmetry about $u = 0$) are shown in Figure 1. We see that the influence of oscillations on the velocity behaviour for the k dependence is monotonically decreasing: the greater the wavenumber k, the smaller the velocity in absolute magnitude (Fig. 1a). This effect may

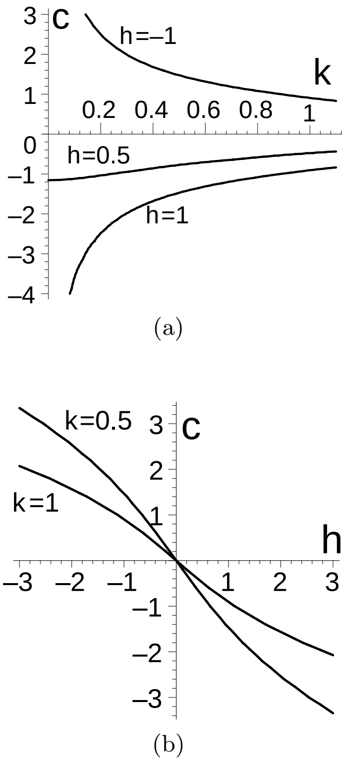

Fig. 1. Velocity behaviour for the one-component model in the case of the symmetric ($u_0 = 0$) null-cline when $\xi_0 = 0$: (a) k-dependence and (b) h-dependence.

be explained using the fact that when oscillation is fast, the system “feels” only its average and then it does not cause a big effect and the velocity tends to zero. When $k \to 0$, the velocity tends to a constant value such that the smaller is the amplitude of the forcing, the smaller is this constant. The h-dependence of the velocity is presented graphically in Figure 1b. The corresponding curve is slightly different from a straight line: the larger the wavenumber k, the closer is the curve to a straight line. This follows directly from the approximation of equation (8) for large k: $c \simeq -2h/k^2$. And once again, as h changes sign, the front velocity reverses its sign as well. It is pertinent to note here that the influence of the oscillations ($k \neq 0$) on the velocity behaviour, as illustrated in Figure 1, is similar to a damping: if we consider a simple damped (at $\xi \to \pm\infty$) forcing with pulse-shaped profile of the piecewise exponential type $\bar{f}(\xi) = h\exp(-k|\xi|)$ instead of oscillating forcing, the k-dependence will show a similar behaviour.

The second particular case (case II) is when $k\xi_0 = \pi/2 + 2\pi n, n = 0, \pm 1, \pm 2, ...$. In this case the R-term vanishes and we have the following velocity equation

$$c + Q(\lambda_1 - \lambda_2) = 0 \quad (9)$$

or, rewritten in the full form,

$$c[c^2k^2 + (k^2+1)^2] - hck\sqrt{c^2+4} = 0\,. \quad (10)$$

[2] Here we must note that ξ_0 may be large but finite, because it is necessary for the construction of the boundary and matching conditions.

Just as equation (8), this can be turned into a cubic equation for c^2, but in distinction from the former equation this one has the solution $c = 0$ (nonmoving front)[3]. This solution is valid at any value of the forcing amplitude h and for the specific choice of the wavenumber $k = (\pi/2 + 2\pi n)/\xi_0$. When the velocity is equal to zero, the "traveling" periodic forcing degenerates into a spatially periodic one. Both other roots may be imaginary or also real in contrast with equation (8). When the wavenumber k or amplitude h becomes negative, there is only the trivial real solution ($c = 0$). Under a simultaneous change of signs of k and h, the equation (10) remains the same. The set of roots is most readily visualized with the aid of specific values of the phase ξ_0 and the wavenumber k. When we choose $\xi_0 = \pi/2$, $k = 1$ and eliminate the solution $c = 0$, equation (10) reduces to $(c^2 + 4) - h\sqrt{c^2 + 4} = 0$. Hence the velocity $c = \pm\sqrt{h^2 - 4}$ (for positive h) tends to infinity as the amplitude of forcing grows and tends to zero when $h \to 2$.

The velocity diagrams $c = c(k)$ and $c = c(h)$ described by equation (6) at $\xi_0 = \pi/2$ are shown in Figure 2. The velocity *versus* wavenumber dependence ($c = c(k)$, Fig. 2a) has an oscillating behaviour around $c = 0$ axis, so that the velocity may now become positive and negative. However, the decreasing (at growing k) behaviour remains the same as in the case I. The velocity *versus* amplitude dependence ($c = c(h)$, Fig. 2b) shows a bifurcation by which a pair of counterpropagating fronts forms. The bifurcation is perfect at $k = 1$ (this case is determined by (10)) and imperfect when the wavenumber is different from but near $k = 1$. The critical value of the amplitude is $h_{\text{crit}} = (k^2 + 1)^2/2k$.

For the nonmoving front the external forcing is only spatially dependent, so that here the pinning effect [13] is reproduced, meaning that under a steady, spatially periodic forcing the velocity of the front may be zero. Pinning is however not perfect at wavenumbers different from $k = 1$ (at $\xi_0 = \pi/2 + 2\pi n$). Figure 2b shows that at $k = 0.8$ the front does not have exactly zero velocity for small h, so it is not strictly pinned but keeps "creeping" at a small velocity instead. When the velocity is equal to zero, the constants $Q = 0$ and $R = h/(k^2 + 1) \neq 0$ and the particular solution of the inhomogeneous equation contains only a cosine part. In the case II there is no situation with constant forcing because $k\xi_0 \neq 0$.

Note that the curve for $h = 3$ in Figure 2a becomes multivalued below $k \approx 1.1$, due to the bifurcation to three solutions (appearing in Fig. 2b at $h = 2$ for $k = 1$ and at $h \approx 3.6$ for $k = 0.8$). Below $k \approx 0.9$ (and above $k \approx 1.1$), it returns to single-valuedness.

In the other particular cases, when $k\xi_0 = \pi + 2\pi n$ (case III) and $k\xi_0 = 3\pi/2 + 2\pi n, n = 0, \pm1, \pm2, ...$ (case IV) the velocity equation remains the same as in the cases I and II, respectively, with the replacement $h \to -h$. Without this replacement, in the case III the velocity changes the sign as compared with the case I. However, the front curves are slightly different in all four cases. The distinctions between them are clearly visible in the phase portraits which are presented in Figure 3. The forcing parameters, h and k, are the same for all four diagrams. The cases I (Fig. 3a) and III (Fig. 3c) have the opposite velocities and the phase diagrams are asymmetric and change from one curve to another under replacement $u \to -u$. The difference takes place also in the front profile. When the velocity is negative (case I), the front $u(\xi)$ moves from $u = 1$ to $u = -1$ (from right to left in the $u - \mathrm{d}u/\mathrm{d}\xi$ or $\xi - u$ planes). Therefore, when the front runs, the oscillation corresponding to the asymmetric loop near $u = 0.5$ in Figure 3a is behind the front. A similar situation holds when the velocity is positive (case III, Fig. 3c). This effect is the same as one in the case of oscillating fronts in the two-component model without forcing [14]. In the cases II (Fig. 3b) and IV (Fig. 3d), the front is nonmoving and the phase diagrams are symmetric (but not similar) about the $0 - \mathrm{d}u/\mathrm{d}\xi$ axis. This means also that the front profile in the $\xi - u$ plane is symmetric under rotation by 180 degrees around the coordinate origin. From all these curves, it will be obvious that the differences in front structure are significant only in the middle interfacial zone of (ξ, u) and when $|\xi|$ is large the curve has just a loop around corresponding fixed point in the phase plane.

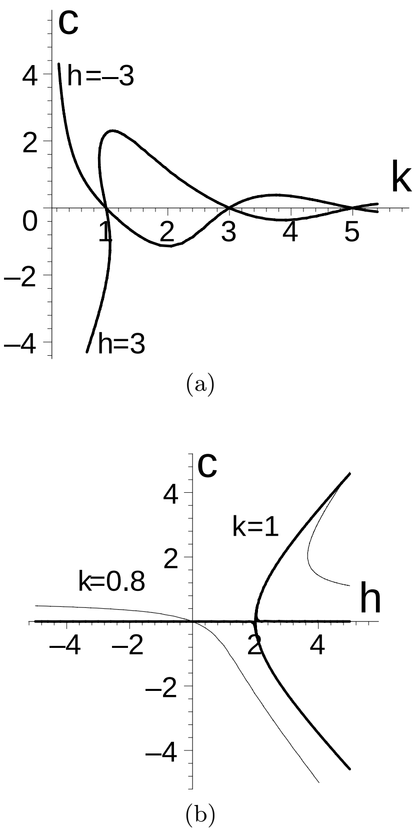

Fig. 2. Velocity behaviour (one-component model with symmetric null-cline) when $\xi_0 = \pi/2$: (a) k-dependence and (b) h-dependences with $k = 1$ and $k = 0.8$ are denoted by thick and thin lines, respectively.

[3] From equation (6) it follows that the front is nonmoving *only when* $\cos(k\xi_0) = 0$ (at $u_0 = 0$).

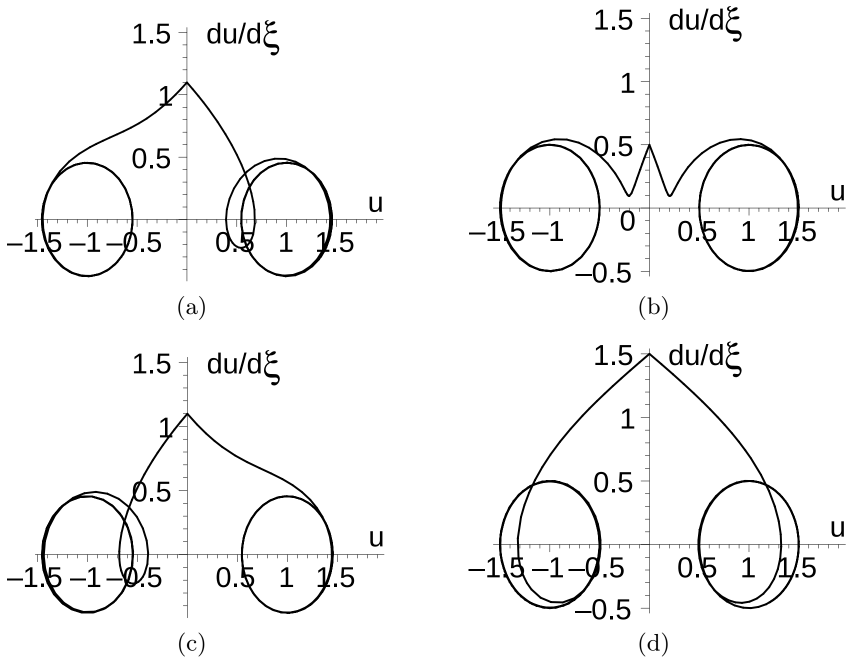

Fig. 3. Phase portrait (one-component model with symmetric null-cline) when (a) $\xi_0 = 0$ ($c \approx -0.9$), (b) $\xi_0 = \pi/2$ ($c = 0$), (c) $\xi_0 = \pi$ ($c \approx 0.9$) and (d) $\xi_0 = 3\pi/2$ ($c = 0$). The forcing parameters are $h = k = 1$.

For the considered one-component system, an adequate explanation of the origin and multiplicity of solutions can be obtained with a known particle-in-a-potential analogy [9]. This analogy is based on the identification of the equation (2) with the equation of motion of a classical particle with friction (c is a friction coefficient) in a potential $\int f(u)\,\mathrm{d}u$ which oscillates with time ξ due to an additional part (forcing) $\bar{f}(\xi)$. One identifies $u(\xi)$ with a spatial coordinate and $f(u)$ with the negative force, derived from the potential. The potential is piecewise parabolic and has a double-hill shape. The front solution $u(\xi)$ is equivalent to the motion of the particle when at time $\xi = \pm\infty$ the particle is located in the maxima at $u = \pm 1$ and at $\xi = \xi_0$ passes through a minimum at $u = u_0$. Using the replacement $\xi \to \xi - \xi_0$ in the equation (2) we find that the particle will pass through the minimum always at $\xi = 0$. However, then the forcing in the model equation (2) becomes ξ_0-dependent $\bar{f}(\xi) = h\cos[k(\xi - \xi_0)]$. Therefore the solution of the inhomogeneous problem is $\bar{u}(\xi) = R\cos[k(\xi - \xi_0)] + Q\sin[k(\xi - \xi_0)]$, *i.e.*, there exists a set of solutions $u(\xi)$ with different phases ξ_0. In the absence of periodic forcing, the ξ_0-dependence vanishes and all solutions are the same.

The appearance of the ξ_0-dependence in the solutions is not a particular feature of the piecewise linear approximation. It is well-known that the solution of the nonlinear equation $u_{\xi\xi} + u - u^3 = 0$ (corresponding to our model at $c = 0$ and $\bar{f}(\xi) = 0$) has a front solution $u(\xi) = \pm\tanh[(\xi - \xi_0)/\sqrt{2}]$, where ξ_0=const. is the center of the wave. Due to translational invariance all solutions with variable ξ_0 are similar. But when this invariance is violated (by inserting $\bar{f}(\xi)$-term in the equation) the solutions become different.

3 Two-component model

In this section, we extend our investigations to the case of a two-component system. We would like to recall here again that two variables are necessary for the oscillatory dynamics in the Belousov-Zhabotinsky reaction. Indeed, as we saw, there are no oscillations in the homogeneous one-component model without external forcing. But in the two-component model it is possible to observe oscillating fronts. The mathematical origin of these oscillations is that there are imaginary values for λ (see Eq. (2.8) in [12]); in the one-component model we have only real λ. Spatial oscillations in the one-component model may exist in hyperbolic reaction-diffusion systems [8]. Thus, the system considered in this section consists of two scalar fields $u(x,t)$ and $v(x,t)$ and is described by the equations

$$\frac{\partial u}{\partial t} = f(u,v) + \bar{f}(x,t) + \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial v}{\partial t} = \varepsilon g(u,v) + \frac{\partial^2 v}{\partial x^2}, \quad (11)$$

with reaction terms $f(u,v) = -\alpha u - 1 + 2\theta(u) - v$ and $g(u,v) = u - v$. Here $\theta(u)$ is the Heaviside function and ε, α are positive constants. This is one of the simplest one-dimensional bistable reaction-diffusion systems involving one activator and one inhibitor. A homogeneous version of this model with nondiffusing inhibitor $v(x,t)$ was used by Rinzel and Keller [15] as the mathematical description of the excitation and propagation of nerve impulses. The case of equal diffusion coefficients considered in our work is also of importance for activator-inhibitor systems, as the diffusion constants are very close to each other for the reactants of widely studied chemical reaction-diffusion systems such as the Belousov-Zhabotinsky reaction [16–18]. The parameter ε is the ratio of the time scales associated with the two fields and has generic significance in that $1/\varepsilon$ measures the ratio of excitation rate to recovery rate. Most studies of the reaction-diffusion systems consider the case $\varepsilon \ll 1$, where the activator concentration $u(x,t)$ is the fast variable and that of the inhibitor the slow one. In this paper we do not impose any restriction on the value of ε.

Now we have two traveling wave equations

$$\begin{aligned} &u_{\xi\xi} + cu_\xi + f(u,v) + \bar{f}(\xi) = 0\,, \\ &v_{\xi\xi} + cv_\xi + \varepsilon g(u,v) = 0 \end{aligned} \tag{12}$$

and two boundary conditions (at $\xi \to \pm\infty$) for the activator field $u(\xi)$ and two conditions for the inhibitor field $v(\xi)$. There are five matching conditions: three equations for the $u(\xi)$-field (with the additional equation) which are the same as in the one-component case and two equations for the $v(\xi)$-field. There is no additional equation for the $v(\xi)$-field, because the matching point in the $u-v$ plane is uniquely determined by fixing the u-coordinate[4]. The general solutions are expressed now as superpositions of four exponentials

$$u(\xi) = \sum_{i=1}^{4} A_i \mathrm{e}^{\lambda_i \xi} + \bar{u}(\xi) + u^*,\; v(\xi) = \sum_{i=1}^{4} B_i \mathrm{e}^{\lambda_i \xi} + \bar{v}(\xi) + v^*, \tag{13}$$

where A_i, B_i are constants, $u^* = v^* = \mp 1/(\alpha+1) = \mp s$=const. The minus and plus signs of s correspond to the first or the second pieces of the front solution, respectively. The constants B_i may be expressed through constants A_i. These expressions will be given below. The four eigenvalues λ_i are

$$\begin{aligned} &\lambda_{1,2} = \\ &-c/2 + \sqrt{c^2/4 + (\varepsilon+\alpha)/2 \pm \sqrt{(\varepsilon+\alpha)^2/4 - \varepsilon(\alpha+1)}}\,, \\ &\lambda_{3,4} = \\ &-c/2 - \sqrt{c^2/4 + (\varepsilon+\alpha)/2 \pm \sqrt{(\varepsilon+\alpha)^2/4 - \varepsilon(\alpha+1)}}\,. \end{aligned} \tag{14}$$

Ito and Ohta [11] obtained an exact solution for a motionless case and a propagating-pulse solution in the large inhibitor diffusion coefficient approximation. We consider here the system with equal diffusion constants. In this situation all solutions are exact.

The front solutions (13) are valid only for some values of ε and α. As indicated above, there exists a range of ε and α where the λ_i become complex: when $\varepsilon_{\rm im}^- < \varepsilon < \varepsilon_{\rm im}^+$, where $\varepsilon_{\rm im}^\pm = \alpha + 2 \pm 2\sqrt{\alpha+1}$, the λ_i have an imaginary part, the solutions $u(\xi)$ and $v(\xi)$ contain cosine and sine terms and the fronts present a damped oscillating behaviour. However, the distinction between monotonic and oscillating fronts for the model without forcing is only in the front profile, also it may be shown that the velocity equation is the same for both front types,[5] and we suppose that the same situation is given in the model with forcing. Therefore, we restrict our consideration here only to the case of real eigenvalues.

Introducing (13) into (12), one can express the constants B_i through A_i: $B_i = (\lambda_i^2 + c\lambda_i - \alpha)A_i, i = 1, ..., 4$. Using (14), we obtain

$$\begin{aligned} B_{1,3} &= [(\varepsilon-\alpha)/2 + \sqrt{(\varepsilon-\alpha)^2/4 - \varepsilon}]A_{1,3}\,, \\ B_{2,4} &= [(\varepsilon-\alpha)/2 - \sqrt{(\varepsilon-\alpha)^2/4 - \varepsilon}]A_{2,4}\,. \end{aligned} \tag{15}$$

Thus, we have 5 unknown constants ($A_1, ..., A_4$ and c) and 5 matching conditions. To obtain the front solutions from two pieces, we take into account the signs of λ_i. We consider the case when ε and α are positive. Hence $\lambda_{1,2} > 0$ and $\lambda_{3,4} < 0$, and front solutions are of the "2+2"-type (a sum of two exponentials at $\xi \leq \xi_0$ is patched together with a sum of two others at $\xi \geq \xi_0$) [6]:

$$\begin{aligned} u_1(\xi) &= A_1 \mathrm{e}^{\lambda_1 \xi} + A_2 \mathrm{e}^{\lambda_2 \xi} + \bar{u}(\xi) - s\,, \\ v_1(\xi) &= B_1 \mathrm{e}^{\lambda_1 \xi} + B_2 \mathrm{e}^{\lambda_2 \xi} + \bar{v}(\xi) - s\,, \end{aligned} \tag{16}$$

$$\begin{aligned} u_2(\xi) &= A_3 \mathrm{e}^{\lambda_3 \xi} + A_4 \mathrm{e}^{\lambda_4 \xi} + \bar{u}(\xi) + s\,, \\ v_2(\xi) &= B_3 \mathrm{e}^{\lambda_3 \xi} + B_4 \mathrm{e}^{\lambda_4 \xi} + \bar{v}(\xi) + s\,. \end{aligned} \tag{17}$$

Thus, the first pieces of these solutions u_1, v_1 contain exponentials that grow with increasing ξ, whereas the second pieces u_2, v_2 contain exponentials decaying with growing ξ. The particular solutions read

$$\begin{aligned} \bar{u}(\xi) &= R_1 \cos(k\xi) + Q_1 \sin(k\xi), \\ \bar{v}(\xi) &= R_2 \cos(k\xi) + Q_2 \sin(k\xi). \end{aligned} \tag{18}$$

The expressions of $R_i, Q_i, i = 1, 2$ are

$$\begin{aligned} R_1 &= h[k_\alpha(k_0^2 + k_\varepsilon^2) + \varepsilon k_\varepsilon]/\Delta, \\ Q_1 &= -hk_0(k_0^2 + k_\varepsilon^2 - \varepsilon)/\Delta, \\ R_2 &= \varepsilon h(k_\alpha k_\varepsilon + \varepsilon - k_0^2)/\Delta, \\ Q_2 &= -\varepsilon hk_0(k_\alpha + k_\varepsilon)/\Delta, \end{aligned} \tag{19}$$

where $k_\alpha = k^2 + \alpha$, $k_\varepsilon = k^2 + \varepsilon$, $k_0 = ck$ and $\Delta = (k_\alpha k_\varepsilon + \varepsilon - k_0^2)^2 + k_0^2(k_\alpha + k_\varepsilon)^2$. Hence, R_1 is always positive when $h > 0$, whereas the other constants may be positive or negative. In the case of constant forcing ($k = 0$), the constants Q_i are equal to zero and $R_1 = R_2 = h$.

The velocity equation is derived from the matching conditions as it was done above in the one-component

[4] The change of the v-coordinate initiates the variation of the speed.

[5] For the homogeneous model this was shown in reference [12].

[6] It is interesting to note that this kind of solutions is still valid at small negative α ($\alpha > -1$). In this case the product $\varepsilon(\alpha+1)$ remains positive and the front solutions are of the "2+2"-type.

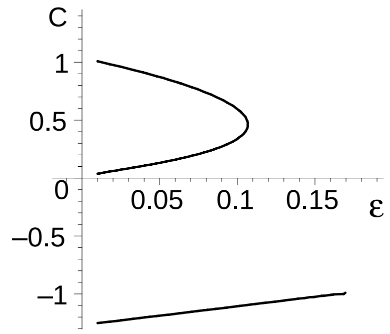

Fig. 4. Bifurcation diagram $c = c(\varepsilon)$ for the two-component model with symmetric null-cline $f(u, v) = 0$ when $\xi_0 = 0$ at $h = 0.1, k = 1$ and $\alpha = 1$.

case. But now this equation is more involved and we do not present the formula here. The resulting equation is so complicated that we do not attempt to solve it analytically – instead, we give, as a figure, some representative solution diagram obtained numerically for specific values of the parameters. The graphical representation of the velocity equation for the case I (when $k\xi_0 = 0$) is illustrated in Figure 4. The velocity diagram $c = c(\varepsilon)$ for fixed null-cline and forcing parameters (h, k and α are fixed) shows a pitchfork bifurcation, which has been referred to in the literature as a non-equilibrium Ising-Bloch bifurcation [19]. We see that under forcing this bifurcation becomes imperfect and the nonmoving front transforms to a moving one. The curve depends, of course, on the forcing parameters: the smaller is the wavenumber k, the more imperfect becomes the bifurcation. For $k \to 0$, there exists a limiting case of imperfect pitchfork at each forcing amplitude $h \neq 0$. A similar imperfect bifurcation takes place in the system without forcing but with non-symmetric ($u_0 \neq 0$) null-cline $f(u, v)$ [12] or in the system with symmetric null-cline and under an external field (as example, an electrical field) [14]. Thus, we can construct our system such that both factors (forcing and field) will approximatively compensate each other.

In Figure 5, we give examples of the front solutions for the case I in the form of the u-v diagram (phase portrait) for a small ratio of the time scales ε (fast activator and slow inhibitor, Fig. 5a) and for large ε (Fig. 5b). In both cases, the other model parameters (h, k and α) are the same. In reference [12], it was shown that in the system without forcing the front curve at large ε tends to the null-cline $g(u, v) = 0$ (in the case of the nonmoving front). Now the phase portraits show that in the case of small ε the line has zig-zag character, whereas at large ε the curve has loops around both fixed points. The origin of this is due to the fact that the oscillations of the inhibitor $v(\xi)$ at large ε are more pronounced, than the ones at small ε. This fact is associated with the expressions of R and Q: the parameters R_2, Q_2 (and hence $v_{1,2}(\xi)$) have a factor ε, whereas R_1, Q_1 (and hence $u_{1,2}(\xi)$) have not (see Eqs. (19)). However, the loop vanishes in the cases II and IV when the front velocity is equal to zero. Thus, the

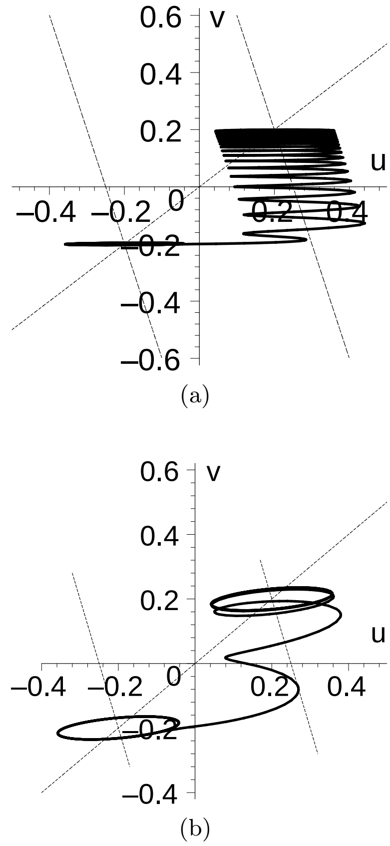

Fig. 5. Front solutions in the form of the u-v diagram (phase portrait) when $\xi_0 = 0$ for (a) small ($\varepsilon = 0.1$) and (b) large ($\varepsilon = 1$) ratio of the time scales ε. The forcing parameters are $h = k = 1$ and the null-cline slope $\alpha = 4$.

nonmoving fronts exist also in the two-component model. At $k\xi_0 = 0$ we have fronts with negative velocities. At other values of $k\xi_0$ the velocity may be positive or zero. A pair of fronts with opposite signs of velocity are arranged symmetrically about the origin in the u-v plane. The front velocity does not vary strongly under changes of the parameter ε (in Figs. 5a and 5b): in case (a) the velocity is $c \approx -3.9$ and in case (b) $c \approx -3.96$. In summary, we would like to note that the $\bar{u}(\xi), \bar{v}(\xi)$ solutions remain the same under the simultaneous changing of signs of the wavenumber k and the velocity c. This is true for all the cases I-IV.

4 Conclusion

In conclusion, we investigated the possibility to control the evolution of front waves by introducing external periodic forcing. Exact analytical solutions for the front propagating under forcing were obtained for one-dimensional piecewise linear reaction-diffusion systems and the corresponding velocity equations were derived. A simple form of the periodic force $\propto \cos(k\xi)$ was used. An analysis of forced reaction-diffusion equations predicts sets of possible solutions (fronts with positive, negative and zero velocities). These solutions are distinguished by their phases (matching point coordinates ξ_0) according to which the velocity *versus* wavenumber dependences (as it was shown for the one-component model) are monotonic or oscillating. For specific wavenumber and phase choice there is a nonmoving front at any value of the forcing amplitude. When the amplitude is large enough the velocity bifurcates to form two counterpropagating fronts. In the two-component case the velocity diagram shows an imperfect Ising-Bloch bifurcation (at zero phase). We investigated the behaviour of the fronts and found that the oscillations due to forcing are present in the phase portrait of the two-component system, too. In the phase diagram there exist loops around both fixed points. For the two-component model these loops vanish (degenerate into lines) when the front velocity is equal to zero or when the model parameter ε is small. In the one-component case the loops don't vanish when the speed is equal to zero.

We do not compare here the oscillations due to forcing and the oscillations in response to system parameters (ε and α), because in the last case the oscillations are damped. However, in the case of a damped periodic forcing $\propto \exp(\mp p\xi)\cos(k\xi)$ it may be shown that the parameter-dependent oscillations have larger wavelength than the forcing oscillations. Near the resonance (when $\lambda \approx p + \mathrm{i}k, \mathrm{i}^2 = -1$), there is no significant front profile change. In the case of the one-component model, it was pointed out that when one considers a simple damping forcing $\propto \exp(\mp\xi)$ (at zero phase) the dependences of the wavenumber on velocity and the amplitude on velocity have similar behaviour as in the case of the oscillating forcing. Therefore, we think that it would be interesting to investigate the system under a δ function forcing, which is a limiting case of a pulse-like force. Due to the δ function in the activator equation the front solution for $u(\xi)$ will have a bend, which originates from a jump in the matching condition for the derivative $\mathrm{d}u(\xi)/\mathrm{d}\xi$.

The external forcing consideration works very well. In reference [20] another type of forcing, presented as spatial inhomogeneities $\bar{f}(x)$, was analyzed to describe a bifurcation of front dynamics. The case of pure temporal forcing, $\bar{f}(t)$ was considered in references [6,21]. However, when traveling wave solutions are examined, it makes no significant difference whether temporal $\bar{f}(t)$ or spatial $\bar{f}(x)$ external forcing is employed, because in both cases we have the same null-cline with time- or space- dependent parameters which indicate zeros of the rate function $f(u)$. Therefore, the "traveling forcing" $\bar{f}(\xi)$ considered in our article becomes a subject of much specific interest.

In this paper, one-dimensional models have been considered. A generalization of these models to the two-dimensional case runs into problems when specifying boundary and matching conditions for fronts in the two-dimensional plane. Solutions of two-dimensional piecewise linear equations do not satisfy the conditions necessary for smooth concatenation [22]. In our opinion, further investigations of a forced reaction-diffusion system may be required towards the modeling of the front dynamics in a medium with different diffusivities of the two components. In the symmetry-breaking spatial differentiation leading to the formation of patterns, the ratio between the diffusion rates of the activator and inhibitor plays an important role [23]. It is known that sharp wavefronts are very stable to lateral distortions, as long as the ratio of diffusion constants of inhibitor to activator is not too large [24]. This case requires more detailed studies and will be discussed elsewhere.

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